Chapter 4

Constrained optimization

Most economic problems consist in finding how to optimally allocate scarce resources. This sentence describes what will be done in this chapter, that is optimizing a function when the decision variables are constrained within limits.

Chapter 3 offered only a few examples because practical problems of finance are constrained optimization problems. However, the "trick" to solve a constrained problem is to transform it in an unconstrained problem having the same solutions, then justifying chapter 3. The price to pay for this transformation is an increase in the number of decision variables.

Of course, to be interesting to study, a constrainted problem should depend at least on two decision variables[1].

Section 4.1 deals with the optimization of functions depending on two variables with one equality constraint. We introduce the Lagrangian in this simple framework. The Lagrangian is the essential tool to solve constrained problems.

Section 4.2 generalizes results of section 4.1 to problems with p variables

[1] With only one variable, two situations are possible: either the solution is in the interior of the domain limited by the constraint or it is on the frontier. In the first case, methods of chapter 3 are still valid, and in the second case, it is enough to compare the values of the function on the frontier to find the optimal one.

Download free eBooks at bookboon.com

and m equality constraints. The final section deals with the most general problem with equality and inequality constraints.

4.1 Functions of two variables and equality constraint

In this section, we focus on the most simple constrained optimization problem (two variables, one constraint). The results are easy to interpret, and their generalization is natural afterwards. This presentation avoids losing the reader into unimportant calculation details.

4.1.1 Problem statement

The objective function f is defined on an open subset $D \subset \mathbb{R}^2$ and is twice continuously differentiable. The constraint is written by means of a function g, defined on D and also twice continuously differentiable. We develop hereafter the case of a maximization problem, but the reasoning is similar for a minimization. The two cases (maximum and minimum) will be separated when necessary.

The optimization problem, denoted $\mathcal{P}$, writes:

$$\begin{aligned} &\max_{(x_1,x_2)\in D} f(x_1,x_2) \\ &\text{u.c. } g(x_1,x_2) = c \end{aligned} \qquad (\mathcal{P})$$

where $c \in \mathbb{R}$ is given[2].

For example, if g is a budget constraint in a utility maximization problem, $g(x_1,x_2) = c$ means $p_1x_1 + p_2x_2 = R$ where $c = R$ is the wealth of the consumer. Such a linear constraint induces an explicit relationship between the two decision variables, that is $x_2 = (R - p_1x_1)/p_2$. In such a simple case,

[2] u.c is a shortcut for "under the constraints"

Download free eBooks at bookboon.com

the constraint writes $x_2 = h(x_1)$ where h is a one-variable function. When this kind of transformation is possible, we are back to the single-variable (unconstrained) problem written as:

$$\max_{x_1} f(x_1, h(x_1))$$

In general, this transformation cannot be used. This is the reason why the Lagrangian has been introduced to solve optimization problems. It is another way to transform a constrained problem into an unconstrained problem (without changing the optimal values of the decision variables).

To illustrate what is going on, denote $x_2^* = \phi(x_1^*)$ where (x_1^*, x_2^*) is a local optimum of f, and x_2^* is an implicit function de x_1^* (see chapter 2), by means of the constraint $g(x_1, x_2) = c$.

The derivation of compound functions can be used to calculate the derivative of $F(x_1) = f(x_1, \phi(x_1))$ at $x_1 = x_1^*$ (chapter 2 of part I). We then write:

$$F'(x_1^*) = \frac{\partial f}{\partial x_1}(x_1^*, \phi(x_1^*)) + \phi'(x_1^*)\frac{\partial f}{\partial x_2}(x_1^*, \phi(x_1^*)) \tag{4.1}$$

The implicit function theorem allows to deduce:

$$\phi'(x_1^*) = -\frac{\frac{\partial g}{\partial x_1}(x^*)}{\frac{\partial g}{\partial x_2}(x^*)} \tag{4.2}$$

At the optimum we know that $F'(x_1^*) = 0$. Equations (4.1) and (4.2) lead to:

$$\frac{\frac{\partial g}{\partial x_1}(x^*)}{\frac{\partial g}{\partial x_2}(x^*)} = \frac{\frac{\partial f}{\partial x_1}(x^*)}{\frac{\partial f}{\partial x_2}(x^*)} \tag{4.3}$$

If λ is defined by:

$$\lambda = \frac{\frac{\partial f}{\partial x_1}(x^*)}{\frac{\partial g}{\partial x_1}(x^*)}$$

Download free eBooks at bookboon.com

we obtain:

$$\frac{\partial f}{\partial x_1}(x^*) - \lambda \frac{\partial g}{\partial x_1}(x^*) = 0 \quad (4.4)$$

$$\frac{\partial f}{\partial x_2}(x^*) - \lambda \frac{\partial g}{\partial x_2}(x^*) = 0 \quad (4.5)$$

Equation (4.5) comes from relation (4.3). More generally, equations (4.4) and (4.5) are useful to specify the intuition behind the definition of the Lagrangian.

4.1.2 Lagrangian and optimality conditions

Definition 142 *The **Lagrangian** of problem $\mathcal{P}$ is the function $\mathcal{L}(\lambda, x)$ defined by:*

$$\mathcal{L}(\lambda, x) = f(x) + \lambda\,(c - g(x))$$

*λ is the **Lagrange multiplier** of the constraint $g(x) = c$.*

Remark 143 *In some books, the right-hand side of the constraint is 0. Of course, defining $g^*(x) = g(x) - c$ leads to write the Lagrangian as $f(x) - \lambda g^*(x)$ and the constraints as $g^*(x) = 0$.*

The following proposition shows how problem $\mathcal{P}$ is solved with the methods developed in chapter 3 by optimizing $\mathcal{L}$. It is worth noticing that if f is a function depending on two variables, $\mathcal{L}$ is a function of three variables.

Solving $\mathcal{P}$ is equivalent to optimize the Lagrangian without constraints (denote $\mathcal{P}'$ this problem):

$$\max_{(\lambda,x)} \mathcal{L}(\lambda, x) \quad (\mathcal{P}')$$

If (λ^*, x^*) is a local optimum of $\mathcal{P}'$, proposition 136 of chapter 3 says that

Download free eBooks at bookboon.com

the partial derivatives of $\mathcal{L}$ are equal to 0 at (λ^*, x^*), that is:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x_1}(\lambda^*, x^*) &= \frac{\partial f}{\partial x_1}(x^*) - \lambda^* \frac{\partial g}{\partial x_1}(x^*) = 0 \\
\frac{\partial \mathcal{L}}{\partial x_2}(\lambda^*, x^*) &= \frac{\partial f}{\partial x_2}(x^*) - \lambda^* \frac{\partial g}{\partial x_2}(x^*) = 0 \\
\frac{\partial \mathcal{L}}{\partial \lambda^*}(\lambda^*, x^*) &= c - g(x^*) = 0
\end{aligned}$$

The last equation $c - g(x^*) = 0$ simply means that the constraint is satisfied. A consequence is that the optimal value of $\mathcal{L}$ is also the optimal value of f. This method kills two birds with one stone. It transforms a difficult problem in an easy one by optimizing another function, but the optimal values are the same in the two problems.

Proposition 144 *If x^* is a local maximum of f under the constraint $g(x) = c$ and if the gradient of g is not zero at x^*, there exists λ^* satisfying:*

$$\frac{\partial \mathcal{L}}{\partial x_i}(\lambda^*, x^*) = \frac{\partial f}{\partial x_i}(x^*) - \lambda^* \frac{\partial g}{\partial x_i}(x^*) = 0$$

for $i = 1, 2$.

Proposition 144 is a necessary optimality condition. x^* must be an optimum for the relationship to be satisfied. As in chapter 3, second-order conditions involving the Hessian matrix of $\mathcal{L}$ are required to obtain sufficient optimality conditions.

The condition on the gradient of g (it should not be zero) is satisfied in most finance problems. In fact, the standard finance problem has linear constraints, either a budget constraint to maximize an expected utility or a portfolio constraint in portfolio choice problems. The gradient of g cannot be 0 when the constraint is linear.

Proposition 145 (λ^*, x^*) *is a local maximum (minimum) of $\mathcal{L}$ if the following conditions are satisfied:*

Download free eBooks at bookboon.com

1) $\nabla \mathcal{L}(\lambda^*, x^*) = 0.$

2) The determinant of $H_{\mathcal{L}}(\lambda^*, x^*)$ *is positive (negative).*

Condition (2) deserves some comments, because only the determinant of $H_{\mathcal{L}}(\lambda^*, x^*)$ seems to be involved, and not all the principal minors, contrary to the sufficient optimality conditions in chapter 3. In fact, the formulation of the condition comes from the structure of $H_{\mathcal{L}}$. $\mathcal{L}$ is a linear function of λ, so the first principal minor of $H_{\mathcal{L}}$ is always 0 because it is equal to the second-order derivative of $\mathcal{L}$ with respect to λ.

The second principal minor, denoted M_2, is equal to:

$$M_2 = \begin{vmatrix} 0 & -\frac{\partial g}{\partial x_1}(x^*) \\ -\frac{\partial g}{\partial x_1}(x^*) & \frac{\partial^2 f}{\partial x_1^2}(x^*) - \lambda^* \frac{\partial^2 g}{\partial x_1^2}(x^*) \end{vmatrix} = -\left(\frac{\partial g}{\partial x_1}(x^*)\right)^2 < 0$$

M_2 is always negative, meaning that the effective optimality condition can only concern the sign of the last principal minor, that is the determinant of $H_{\mathcal{L}}(\lambda^*, x^*)$.

Notice that proposition 144 includes a condition on the gradient of g. This condition appears here in part (2) of proposition 145. In fact, if the gradient of g was 0, the first line of $H_{\mathcal{L}}(x^*)$ would be null and $\det(H_{\mathcal{L}}(x^*))$ would also be equal to 0.

Download free eBooks at bookboon.com
Click on the ad to read more

Example 146 *Consider the utility maximization problem under a budget constraint (the notations are as usual):*

$$\begin{aligned}\max_{(x_1,x_2)\in\mathbb{R}^*} U(x_1,x_2) &= \sqrt{x_1x_2} \\ u.c.\ p_1x_1+p_2x_2 &= R\end{aligned}$$

The Lagrangian of the problem is:

$$\mathcal{L}(\lambda,x)=U(x_1,x_2)+\lambda(R-p_1x_1-p_2x_2) \tag{4.6}$$

The first-order conditions are the following:

$$\begin{aligned}\frac{\partial U}{\partial x_1}(x^*)-\lambda^*p_1 &= 0\\ \frac{\partial U}{\partial x_2}(x^*)-\lambda^*p_2 &= 0\\ R-p_1x_1^*+p_2x_2^* &= 0\end{aligned}$$

Replacing U by its definition leads to:

$$\begin{aligned}\frac{1}{2}\sqrt{\frac{x_2^*}{x_1^*}}-\lambda^*p_1 &= 0\\ \frac{1}{2}\sqrt{\frac{x_1^*}{x_2^*}}-\lambda^*p_2 &= 0\\ R-p_1x_1^*+p_2x_2^* &= 0\end{aligned}$$

We are back to the standard result of microeconomics. The ratio of marginal utilities is equal to the ratio of prices.

Consider the following parameters, $R=10$; $p_1=3$; $p_2=4$. We obtain

Download free eBooks at bookboon.com

the following conditions:

$$\begin{aligned} \frac{x_2^*}{x_1^*} &= \frac{3}{4} \\ 3x_1^* + 4x_2^* &= 10 \end{aligned} \tag{4.7}$$

meaning that $x_2^* = \frac{5}{4}$ *and* $x_1^* = \frac{5}{3}$.

First, we observe that R *is equally shared between the two goods because* $3 \times \frac{5}{3} = 4 \times \frac{5}{4} = 5$. *This result is in line with intuition. The utility function is symmetric, so the optimal amounts spent in each good are equal.*

Second, the Lagrange multiplier is equal to:

$$\lambda^* = \frac{1}{2p_1}\sqrt{\frac{x_2^*}{x_1^*}} = \frac{\sqrt{3}}{12} = 0.144$$

and the utility at x^* *is* $\sqrt{\frac{5}{4} \times \frac{5}{3}} = 1.4434$

Imagine now that system (4.7) is solved twice, first with $R = 9.8$ *and second with* $R = 10.2$.

If $R = 9.8$, *we obtain:*

$$R = 9.8 \ ; \ x_1^* = \frac{4.9}{3} \ ; \ x_2^* = \frac{9.8}{8} \ ; \ U(x_1^*, x_2^*) = \sqrt{\frac{4.9}{3} \times \frac{9.8}{8}} = 1.4145$$

If $R = 10.2$ *the results are:*

$$R = 10.2 \ ; \ x_1^* = \frac{5.1}{3} \ ; \ x_2^* = \frac{10.2}{8} \ ; \ U(x_1^*, x_2^*) = \sqrt{\frac{5.1}{3} \times \frac{10.2}{8}} = 1.4722$$

The objective function decreases by 0.0289 when R decreases by 0.2 units. A linear approximation gives a decrease in utility of 0.1445 for one unit less in the budget constraint. Symetrically, if R increases by 0.2, utility increases by 0.0288, that is an increase of 0.144 for one more unit spent. 0.144 is exactly the value of the Lagrange multiplier. It is the reason why the Lagrange multiplier measures the sensitivity of utility (objective function) with respect

Download free eBooks at bookboon.com

to variations in available wealth (constraint). The other more direct route to come to this interpretation is to verify that the derivative of the Lagrangian with respect to wealth is exactly λ.

It remains to check that our solution is a maximum. The Hessian matrix of the Lagrangian is:

$$H_{\mathcal{L}}\left(\lambda^*, x^*\right) = \begin{pmatrix} 0 & -1 & -1 \\ -1 & -\frac{1}{4x_1^*}\sqrt{\frac{x_2^*}{x_1^*}} & -\frac{1}{4}\sqrt{\frac{1}{x_1^* x_2^*}} \\ -1 & -\frac{1}{4}\sqrt{\frac{1}{x_1^* x_2^*}} & -\frac{1}{4x_2^*}\sqrt{\frac{x_1^*}{x_2^*}} \end{pmatrix}$$

A few calculations lead to:

$$\begin{aligned} \det(H_{\mathcal{L}}\left(\lambda^*, x^*\right)) &= \frac{1}{4}\left(\frac{\sqrt[4]{x_1^*}}{\sqrt[4]{\left(x_2^*\right)^3}} - \frac{\sqrt[4]{x_2^*}}{\sqrt[4]{\left(x_1^*\right)^3}}\right)^2 \\ &= \frac{1}{4}\frac{\left(x_1^* - x_2^*\right)^2}{\sqrt{\left(x_1^* x_2^*\right)^3}} > 0 \end{aligned}$$

This determinant is positive. x^* *then maximizes* U *under the budget constraint.*

Example 146 is a specific case of the following proposition.

Proposition 147 *If a* C^2-*function* f, *defined on an open convex subset* $D \subset \mathbb{R}^2$, *is concave (convex) and if the constraint* g *is affine on* D, *then any local maximum (minimum) is a global maximum (minimum).*

4.2 Functions of p variables with m equality constraints

We consider now twice continuously differentiable functions $f, g_1, ..., g_m$ defined on an open domain $D \subset \mathbb{R}^p$ and taking their values in $\mathbb{R}$. We also

Download free eBooks at bookboon.com

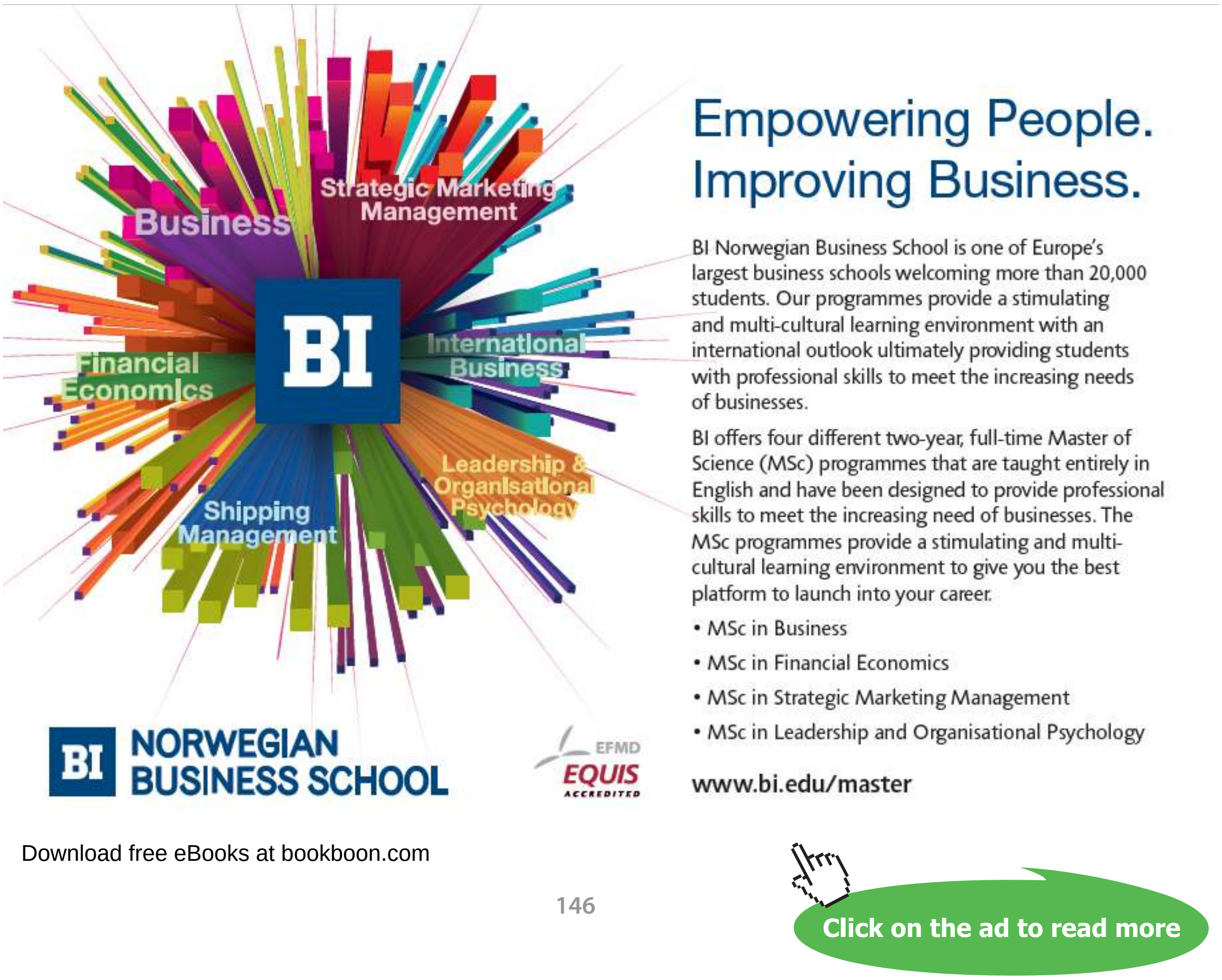
assume $m < p$. The optimization problem addressed in this section is:

$$\max_{x \in D} f(x) \qquad (\mathcal{P})$$
$$\text{u.c. } \; g_j(x) = c_j, \; j = 1, ..., m$$

Following the approach of the preceding section, the Lagrangian of the problem is:

$$\mathcal{L}(\lambda, x) = f(x) + \sum_{j=1}^{m} \lambda_j \left(c_j - g_j(x)\right)$$

There exists one Lagrange multiplier per constraint; the initial problem with p variables and m constraints has become an unconstrained maximization problem with $p + m$ variables.



4.2.1 Local optimality conditions

Necessary conditions

We follow the same structure as before and start with a necessary optimality condition in the following proposition.

Proposition 148 *Let x^* be a local optimum of f, satisfying the constraints of problem $(\mathcal{P})$ and such that the gradients $\nabla g_j(x^*), j = 1, ..., m$ are linearly independent vectors in $\mathbb{R}^p$.*

There exists $\lambda^* \in \mathbb{R}^m$ such that the gradient of $\mathcal{L}$ is the null vector at x^*, that is:

$$\begin{aligned} \frac{\partial f}{\partial x_i}(x^*) - \sum_{j=1}^{m} \lambda_j^* \frac{\partial g_j}{\partial x_i}(x^*) &= 0 \text{ if } i = 1, ..., p \\ c_j - g_j(x^*) &= 0 \text{ if } j = 1, ..., m \end{aligned}$$

Why should the gradients be linearly independent? This condition is not intuitive at all. Consider the following example with three variables and two constraints defined as follows:

$$\begin{aligned} x_1 + 2x_2 + x_3 &= c_1 \\ 2x_1 + 4x_2 + 2x_3 &= c_2 \end{aligned}$$

The left hand side of the second equality is twice the left hand side of the first one. Therefore, we can face two situations. If $c_2 \neq 2c_1$, the problem has no solution. But if $c_1 = 2c_2$, the two constraints are redundant, one is enough. The gradients are equal to:

$$\nabla g_1(x) = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \text{ et } \nabla g_2(x) = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}$$

These two vectors are colinear because $\nabla g_2(x) = 2 \times \nabla g_1(x)$.

Download free eBooks at bookboon.com

In general when the gradients are not linearly independent, at least one constraint can be removed before applying proposition 148.

In problems with constraints, Lagrange multipliers are interpreted as in single-variable problems. Each multiplier measures the sensitivity of the objective function with respect to variations in the right-hand side of the constraints. These multipliers lose their significance when gradients are colinear. The problem has the same nature as the one of multicolinearity in multiple regression. When independent variables are colinear, nothing relevant can be said about the significance of the regression coefficients.

Sufficient optimality conditions

After reading chapter 3, the reader knows that sufficient optimality conditions are based on the Hessian matrix of the Lagrangian. However, this matrix is really special because $\mathcal{L}$ is a linear function of the multipliers λ_j. Therefore the second-order derivatives with respect to the multipliers λ_j are 0. In a problem with m constraints, the (m,m)-dimensional North-West corner of $H_{\mathcal{L}}(x)$ only contains zeros. For example, in a problem with 3 variables and two constraints, $H_{\mathcal{L}}(x)$ is as follows:

$$H_{\mathcal{L}}(x) = \begin{bmatrix} 0 & 0 & -\frac{\partial g_1}{\partial x_1}(x) & \frac{\partial g_1}{\partial x_2}(x) & \frac{\partial g_1}{\partial x_3}(x) \\ 0 & 0 & -\frac{\partial g_2}{\partial x_1}(x) & \frac{\partial g_2}{\partial x_2}(x) & \frac{\partial g_2}{\partial x_3}(x) \\ -\frac{\partial g_1}{\partial x_1}(x) & -\frac{\partial g_2}{\partial x_1}(x) & \frac{\partial^2 \mathcal{L}}{\partial x_1^2}(x) & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2}(x) & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_3}(x) \\ -\frac{\partial g_1}{\partial x_2}(x) & -\frac{\partial g_2}{\partial x_2}(x) & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2}(x) & \frac{\partial^2 \mathcal{L}}{\partial x_2^2}(x) & \frac{\partial^2 \mathcal{L}}{\partial x_3 \partial x_2}(x) \\ -\frac{\partial g_1}{\partial x_3}(x) & -\frac{\partial g_2}{\partial x_3}(x) & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_3}(x) & \frac{\partial^2 \mathcal{L}}{\partial x_3 \partial x_2}(x) & \frac{\partial^2 \mathcal{L}}{\partial x_3^2}(x) \end{bmatrix} = \begin{bmatrix} H_1 & H_2' \\ H_2 & H_3 \end{bmatrix}$$

H_1 is a $(2,2)$ null matrix. H_2 is a $(3,2)$ matrix containing the derivatives of the constraints with respect to the variables and H_3 is a $(3,3)$ matrix the elements of which are the second-order derivatives of $\mathcal{L}$ with respect to the three variables.

The structure of $H_{\mathcal{L}}(x)$ implies that the first $2m$ principal minors are not

Download free eBooks at bookboon.com

significant in characterizing the optimum. In general, if the problem has p variables and m constraints, only the sign of the $p-m$ last principal minors matter. In fact, $H_{\mathcal{L}}$ is $(p+m, p+m)$-dimensional and, as just described, the $2m$ first principal minors are not significant. The number of significant minors is then $p+m-2m = p-m$.

Proposition 149 *x^* is a local maximum of f if:*

1) The constraints are satisfied at x^*.

2) There exists a vector of multipliers λ^* satisfying $\nabla\mathcal{L}(\lambda^*, x^*) = 0$.

3) The signs of the last $p-m$ principal minors of $H_{\mathcal{L}}(\lambda^*, x^*)$ alternate, the first one being negative if m is even and positive if m is odd.

Part (3) means that, if the condition is satisfied, the Hessian matrix is negative semi-definite.

Proposition 150 *x^* is a local minimum of f if:*

1) The constraints are satisfied at x^*.

2) There exists a vector of multipliers λ^* satisfying $\nabla\mathcal{L}(\lambda^*, x^*) = 0$.

3) The last $p-m$ principal minors of $H_{\mathcal{L}}(\lambda^*, x^*)$ have the same sign as $(-1)^m$.

Part (3) means that, if the condition is satisfied, the Hessian matrix is positive semi-definite.

4.2.2 Global optimality conditions

The global optimality conditions are quite close to the conditions proposed for functions depending on two variables. The difference comes from the existence of multiple constraints. It is the reason why we do not comment this proposition. The reasoning used for functions of two variables is still valid here.

Download free eBooks at bookboon.com

Proposition 151 *If f, defined on an open convex set $D \subset \mathbb{R}^p$, is concave (convex), and if the constraints g_j are affine functions on D, any local maximum(minimum) is also global.*

We can now address the general case in which the two types of constraints (equalities and inequalities) coexist.

4.3 Functions of p variables with mixed constraints

4.3.1 The problem

This last section addresses the most general problem where inequality and equality constraints coexist. The functions f, g_j, h_k of problem (4.8) are defined on an open subset D in $\mathbb{R}^p$ and twice continuously differentiable. The optimization problem writes:

$$\begin{aligned} \max f(x) \\ \text{u.c. } g_j(x) = c_j, j = 1, ..., m \\ h_k(x) \leq b_k, k = 1, ..., n \end{aligned} \tag{4.8}$$

Remark 152 *Choosing inequality constraints as « $\leq$ » does not matter because any inequality $h(x) \geq c$ is equivalent to $-h(x) \leq -c$.*

To gain in clarity when stating the optimality conditions we used different notations depending on the type of constraint (h for inequalities and g for equalities).

To emphasize the link with the results of the preceding section, imagine that a solution x^* to problem (4.8) has been found. The set of inequality constraints may be divided in two subsets: the first subset contains the constraints satisfying $h_k(x^*) = b_k$, and the second subset contains the con-

Download free eBooks at bookboon.com

straints satisfying $h_k(x^*) < b_k$. The following definition specifies the concept of binding constraints.

Definition 153 *A constraint k is* ***binding*** *at x^* if $h_k(x^*) = b_k$.*

As mentioned in the introduction of the chapter, the constraints often refer to scarce resources. b_k then denotes the quantity of the available resource and the equality $h_k(x^*) = b_k$ means that all the resource has been consumed at x^*. It explains why the word *binding* is used.

4.3.2 The solution

Finding a solution may be difficult because, at an optimum x^* some constraints may be binding and other ones not binding. We usually interpret Lagrange multipliers as measures of the sensitivity of the objective function to variations of the right-hand side of the constraint. But in this approach, the multiplier of a constraint should be 0 when a constraint is not binding. In fact, consider the standard economic problem of utility maximization under a budget constraint, but assume that the utility function is not strictly increasing[3]. It may happen that a part of the budget is not "consumed" at the optimum x^* because the marginal utility is 0 at x^*. In this situation, one more unit of wealth would not increase utility and the multiplier would be 0.

The Lagrangian of problem (4.8) is:

$$\mathcal{L}(\lambda, \mu, x) = f(x) + \sum_{j=1}^{m} \lambda_j \left(c_j - g_j(x)\right) + \sum_{k=1}^{n} \mu_k \left(b_k - h_k(x)\right)$$

If x^* is an optimum for multipliers $\lambda^* = (\lambda_1^*, ..., \lambda_m^*)$ and $\mu^* = (\mu_1^*, ..., \mu_n^*)$,

[3] In many restaurants, the quantity of soft drinks (or sometimes appetizers) you can consume is unlimited. The reason is simply that the optimal choice of a client is not to drink an unlimited quantity of soda. The utility function for soda cannot be strictly increasing everywhere.

Download free eBooks at bookboon.com

then:

$$\begin{aligned}\mu_k &> 0 \text{ and } b_k - h_k(x) = 0 \text{ if constraint } k \text{ is binding.}\\ \mu_k &= 0 \text{ and } b_k - h_k(x) > 0 \text{ if constraint } k \text{ is not binding.}\end{aligned}$$

In the two cases the product $\mu_k (b_k - h_k(x))$ is equal to zero. This remark is used to shorten the formulation of optimality conditions. The coefficients μ_k are called **Kuhn-Tucker multipliers.**

4.3.3 Necessary optimality condition

Proposition 154 *If x^* is a local maximum of f in problem 4.8 and if the gradients of all functions g_j and h_k for which $h_k(x^*) = 0$ are linearly independent, there exist $m + n$ numbers $\lambda_1^*, ..., \lambda_m^*, \mu_1^*, ..., \mu_p^*$ satisfying the three following conditions:*

$$\nabla f(x^*) - \sum_{j=1}^{m} \lambda_j^* \nabla g_j(x^*) - \sum_{k=1}^{n} \mu_k^* \nabla h_k(x^*) = 0$$
$$\forall k \in \{1, 2, ..., n\}, \mu_k^*(b_k - h_k(x^*)) = 0$$
$$\forall k \in \{1, 2, ..., p\}, \lambda_k^* \geq 0$$

In this proposition, the multipliers μ_k^* are positive or equal to 0. In fact, if a constraint is binding (think to these constraints as limitations for some resources), it means that all the resource is consumed at the optimum x^*. Obtaining one more unit of the resource would improve the optimal value of the objective function. μ_k^* then measures the variation of the objective function that would arise if one more unit of the resource k was made available.

Proposition 154 could be written for a minimization problem by simply changing the sign of coefficients μ_k^*.

Download free eBooks at bookboon.com

4.3.4 Necessary and sufficient global optimality conditions

In the preceding section we obtained a global maximum if D is convex, f concave and the functions g_j affine. When the problem includes inequality constraints,this result is generalized as follows.

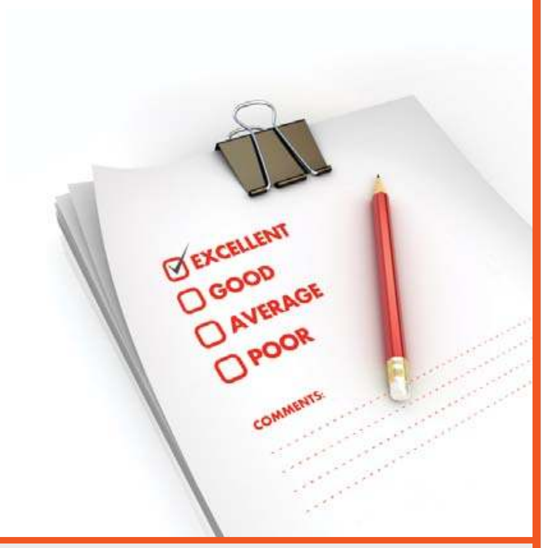
Proposition 155 *If f is concave, the functions g_j affine and the functions h_k convex, the conditions of proposition 154 mean that x^* is a global maximum of f under the constraints of problem (4.8).*

If the problem is a minimization problem, replace « f concave » by « f convex » and change the signe of the coefficients μ_k^* (they would be negative).

Need help with your dissertation?

Get in-depth feedback & advice from experts in your topic area. Find out what you can do to improve the quality of your dissertation!

Get Help Now



Go to www.helpmyassignment.co.uk for more info

Helpmyassignment

Download free eBooks at bookboon.com

Click on the ad to read more